

QUANTUM PRINCIPAL BUNDLES AS HOPF-GALOIS EXTENSIONS

MICO ĐURĐEVIĆ

ABSTRACT. It is shown that every quantum principal bundle with a compact structure group is a Hopf-Galois extension. This property naturally extends to the level of general differential structures, so that every differential calculus over a quantum principal bundle with a compact structure group is a graded-differential variant of the Hopf-Galois extension.

1. INTRODUCTION

The aim of this letter is to establish some connections between quantum principal bundles and Hopf-Galois extensions. In the framework of non-commutative differential geometry [3], the Hopf-Galois extensions [8] are understandable as analogs of principal bundles. If M is a classical smooth manifold, G a Lie group and P a classical principal G -bundle over M , then the right action of G on P induces a diffeomorphism

$$P \times G \leftrightarrow P \times_M P \quad (p, g) \mapsto (p, pg).$$

The freeness of the action of the structure group is equivalent to the injectivity of the introduced canonical map.

Incorporating the above diffeomorphism property to non-commutative geometry, where M and P are general quantum spaces (described by noncommutative algebras) and G is a quantum group (Hopf algebra), leads to the concept of a Hopf-Galois extension.

A more general approach to defining noncommutative-geometric concept of a principal bundle is to generalize only the idea that G acts freely on P . Such *quantum principal bundles* [5] trivially include Hopf-Galois extensions, but the converse is generally not true.

In the next section we show that if G is a compact matrix quantum group [9], then every quantum principal G -bundle P , defined as a quantum space on which the structure group acts freely on the right, is actually a Hopf-Galois extension.

In Section 3 the same question is analyzed for differential calculi over quantum principal bundles. It turns out that every differential structure [5] over a quantum principal bundle P is understandable as a Hopf-Galois extension, at the level of graded-differential algebras [7].

Finally, in Section 4 some concluding remarks are made.

2. THE LEVEL OF SPACES

Let G be a compact matrix quantum group, represented by a Hopf $*$ -algebra \mathcal{A} , the elements of which play the role of polynomial functions on G . The co-product, counit and the antipode will be denoted by ϕ , ϵ and κ respectively. Let $\text{Rep}(G)$ be the category of finite-dimensional unitary representations of G . For each $u \in \text{Rep}(G)$ we shall denote by H_u the carrier unitary space, so that we have the comodule structure map $u: H_u \rightarrow H_u \otimes \mathcal{A}$. We shall denote by \times the product in $\text{Rep}(G)$. Let \mathcal{T} be the complete set of mutually non-equivalent irreducible representations of G . Let \emptyset be the trivial representation of G , acting in $H_\emptyset = \mathbb{C}$.

Let M be a quantum space, represented by a $*$ -algebra \mathcal{V} . Let $P = (\mathcal{B}, i, F)$ be a quantum principal G -bundle over M . By definition [5], the map $F: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ is a $*$ -homomorphism satisfying

$$(\text{id} \otimes \phi)F = (F \otimes \text{id})F \quad (\text{id} \otimes \epsilon)F = \text{id}.$$

It corresponds to the dualized right action of G on P . The map $i: \mathcal{V} \rightarrow \mathcal{B}$ is a $*$ -monomorphism representing the projection map. The image $i(\mathcal{V}) \subseteq \mathcal{B}$ coincides with the F -fixed point subalgebra of \mathcal{B} . Geometrically this means that M is identifiable, via the projection, with the corresponding orbit space. The final condition corresponds to the requirement that G acts freely on P . We require that for each $a \in \mathcal{A}$ there exists elements $q_k, b_k \in \mathcal{B}$ such that

$$\sum_k q_k F(b_k) = 1 \otimes a.$$

Equivalently, the map $X: \mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ given by $X(q \otimes b) = qF(b)$ is surjective. This definition implies that the domain of X can be factorized to the tensor product over \mathcal{V} , which will be denoted by \otimes_M . We shall denote by the same symbol the projected map $X: \mathcal{B} \otimes_M \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$.

In this section we are going to prove that X is bijective. In other words, this means that every quantum principal bundle P is undersandable as a Hopf-Galois extension. The proof will be based on the representation theory [9] of compact matrix quantum groups. We shall construct explicitly the inverse of X .

For each $u \in \text{Rep}(G)$ let \mathcal{E}_u be the space of intertwiners between u and F . These spaces are \mathcal{V} -bimodules, finite and projective on both sides. We have $\mathcal{E}_\emptyset = \mathcal{V}$, in a natural manner.

The following natural decomposition holds

$$\mathcal{B} = \sum_{\alpha \in \mathcal{T}}^{\oplus} \mathcal{B}^\alpha, \quad \mathcal{B}^\alpha = \mathcal{E}_\alpha \otimes H_\alpha.$$

The above decomposition is specified by $\varphi(x) \leftrightarrow \varphi \otimes x$.

As explained in [6], the product map in \mathcal{B} induces a natural identification

$$\mathcal{E}_u \otimes_M \mathcal{E}_v \leftrightarrow \mathcal{E}_{u \times v}$$

for each $u, v \in \text{Rep}(G)$. Furthermore, every intertwiner $f \in \text{Mor}(u, v)$ induces, via the composition map, a bimodule homomorphism $f_\star: \mathcal{E}_v \rightarrow \mathcal{E}_u$, so that we have a contravariant functor from $\text{Rep}(G)$ to the category of finite projective bimodules. In particular, the contraction maps $\check{\jmath}^u: H_u^* \otimes H_u \rightarrow \mathbb{C}$ induce bimodule injections

$\mathfrak{Y}_*^u: \mathcal{V} \rightarrow \mathcal{E}_u^* \otimes_M \mathcal{E}_u$, where we have identified $\mathcal{E}_{u^c} \leftrightarrow \mathcal{E}_u^*$ with the help of the $*$ -structure in \mathcal{B} . Here $u^c: H_u^* \rightarrow H_u^* \otimes \mathcal{A}$ and \mathcal{E}_u^* are the corresponding conjugate representation and the conjugate module.

By construction, we have

$$(1) \quad \mathfrak{Y}_*^u(1) = \sum_k \nu_k \otimes \mu_k \quad \sum_k \nu_k(e_i^*) \mu_k(e_j) = \delta_{ij} 1,$$

where $\{e_i\}$ is an orthonormal basis in H_u , and $\{e_i^*\}$ the corresponding biorthogonal basis in H_u^* .

Let $\tau: \mathcal{A} \rightarrow \mathcal{B} \otimes_M \mathcal{B}$ be a linear map defined by

$$(2) \quad \tau(u_{ij}) = \sum_k \nu_k(e_i^*) \otimes \mu_k(e_j),$$

where $u \in \mathcal{T}$. This map can be naturally extended to $\tau: \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{B} \otimes_M \mathcal{B}$, by imposing the left \mathcal{B} -linearity.

Proposition 1. *The maps τ and X are mutually inverse.*

Proof. A direct computation gives

$$X\tau(b \otimes u_{ij}) = b \sum_k X(\nu_k(e_i^*) \otimes \mu_k(e_j)) = b \sum_{kn} \nu_k(e_i^*) \mu_k(e_n) \otimes u_{nj} = b \otimes u_{ij},$$

where we have used (1) and the intertwining property. Similarly, if $\mu \in \mathcal{E}_u$ then

$$\begin{aligned} \tau X(b \otimes \mu(e_i)) &= \sum_j \tau(b \mu(e_j) \otimes u_{ji}) = \sum_{kj} b \mu(e_j) \nu_k(e_j^*) \otimes \mu_k(e_i) \\ &= \sum_{kj} b \otimes \mu(e_j) \nu_k(e_j^*) \mu_k(e_i) = b \otimes \mu(e_i), \end{aligned}$$

because of the F -invariance of $\sum_j \mu(e_j) \nu_k(e_j^*)$.

In conclusion, X is bijective and $X^{-1} = \tau$. \square

It is worth noticing that if the structure group is non-compact, there exists a variety of quantum principal bundles which are not Hopf-Galois extensions (the map X has a non-trivial kernel).

3. THE LEVEL OF DIFFERENTIAL STRUCTURES

In this section we shall consider general differential structures on quantum principal bundles, and prove that they are always understandable as Hopf-Galois extensions for differential algebras. Considerations of this section are based on the general theory of differential forms and connections developed in [5].

3.1. Preliminaries About Differential Calculi

Let Γ be a bicovariant $*$ -calculus [10] over G . Let Γ_{inv} be the space of left-invariant elements of Γ . There exists the canonical projection map $\pi: \mathcal{A} \rightarrow \Gamma_{inv}$, given by the formula

$$\pi(a) = \kappa(a^{(1)})d(a^{(2)}).$$

The group G naturally acts on the space Γ_{inv} , via the projection $\varpi: \Gamma_{inv} \rightarrow \Gamma_{inv} \otimes \mathcal{A}$ of the adjoint action. Explicitly,

$$\varpi\pi = (\pi \otimes \text{id})\text{ad}, \quad \text{ad}(a) = a^{(2)} \otimes \kappa(a^{(1)})a^{(3)}.$$

Let us assume that the higher-order calculus on G is described by the corresponding [4] universal envelope Γ^\wedge . Let Γ^\otimes be the tensor bundle algebra over Γ . Let us denote by $\Gamma_{inv}^{\wedge, \otimes}$ the corresponding left-invariant subalgebras. These algebras are $*$ -invariant. The action ϖ admits natural extensions to $\Gamma_{inv}^{\wedge, \otimes}$.

Let us consider a graded-differential algebra $\Omega(P)$, representing the calculus on P . By definition, this means that $\Omega^0(P) = \mathcal{B}$, and that \mathcal{B} generates the whole differential algebra $\Omega(P)$, as well as that F is extendible (necessarily uniquely) to a homomorphism $\widehat{F}: \Omega(P) \rightarrow \Omega(P) \widehat{\otimes} \Gamma^\wedge$ of graded-differential $*$ -algebras. Geometrically, the map \widehat{F} corresponds to the “pull back” of the right action map.

Let $\mathfrak{hor}(P) \subseteq \Omega(P)$ be a $*$ -subalgebra representing horizontal forms. By definition,

$$\mathfrak{hor}(P) = \widehat{F}^{-1}(\Omega(P) \otimes \mathcal{A}).$$

We have

$$F^\wedge(\mathfrak{hor}(P)) \subseteq \mathfrak{hor}(P) \otimes \mathcal{A},$$

where $F^\wedge: \Omega(P) \rightarrow \Omega(P) \otimes \mathcal{A}$ is the corresponding right action of G , defined as a composition $F^\wedge = (\text{id} \otimes p_*)\widehat{F}$, and $p_*: \Gamma^\wedge \rightarrow \mathcal{A}$ is the projection map.

Let $\Omega(M) \subseteq \mathfrak{hor}(P)$ be a (graded-differential) $*$ -subalgebra consisting of F^\wedge -invariant elements. Equivalently, $\Omega(M)$ is a \widehat{F} -fixed point subalgebra of $\Omega(P)$. It represents the calculus on M .

By definition, a connection on the bundle P is every first-order hermitian linear map $\omega: \Gamma_{inv} \rightarrow \Omega(P)$ satisfying

$$\widehat{F}\omega(\vartheta) = \sum_k \omega(\vartheta_k) \otimes c_k + 1 \otimes \vartheta$$

where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$. As explained in [5], every quantum principal bundle P admits a connection.

Let us fix a splitting of the form $\Gamma_{inv}^\otimes = S_{inv}^\wedge \oplus \Gamma_{inv}^\wedge$, realized via hermitian grade-preserving right-covariant section $\iota: \Gamma_{inv}^\wedge \rightarrow \Gamma_{inv}^\otimes$ of the corresponding factor-projection map. Here $S^\wedge \subseteq \Gamma^\otimes$ is the quadratic ideal defining Γ^\wedge , and $S_{inv}^\wedge \subseteq \Gamma_{inv}^\otimes$ is its left-invariant part.

Let $\omega^\wedge: \Gamma_{inv}^\wedge \rightarrow \Omega(P)$ be the restriction of the unital multiplicative extension ω^\otimes to Γ_{inv}^\wedge . The following natural decompositions hold

$$(3) \quad m_\omega: \mathfrak{hor}(P) \otimes \Gamma_{inv}^\wedge \leftrightarrow \Omega(P)$$

$$(4) \quad m_\omega^-: \Gamma_{inv}^\wedge \otimes \mathfrak{hor}(P) \leftrightarrow \Omega(P),$$

where m_ω , and m_ω^- are the unique left/right $\mathfrak{hor}(P)$ -linear extensions of ω^\wedge .

3.2. Hopf-Galois Extensions

The map X admits a natural extension to $\widehat{X}: \Omega(M) \otimes_M \Omega(P) \rightarrow \Omega(P) \widehat{\otimes} \Gamma^\wedge$, explicitly

$$\widehat{X}(\alpha \otimes \beta) = \alpha \widehat{F}(\beta).$$

In this subsection, the symbol \otimes_M will be used for the tensor product over $\Omega(M)$.

Proposition 2. *The map \widehat{X} is bijective.*

Proof. The restriction map

$$(5) \quad \widehat{X}: \Omega(P) \otimes_M \mathfrak{hor}(P) \rightarrow \Omega(P) \otimes \mathcal{A}$$

is bijective, as directly follows from the decomposition (4) and the proof of Proposition 1.

We shall first prove that all spaces $\Omega(P) \widehat{\otimes} \Gamma_k^\wedge$ are included in the image of \widehat{X} , using the induction on k . Here $\Gamma_k^\wedge = \sum_{*}^{\oplus} \Gamma^{\wedge j}$, where the summation is performed over $j \leq k$. Let us fix a connection ω on P .

Let us assume that the statement holds for some k . Then for each $\vartheta \in \Gamma_{inv}^{\wedge k+1}$ and $\alpha \in \Omega(P) \otimes_M \mathfrak{hor}(P)$ we have

$$\widehat{X}(\alpha \omega^\wedge(\vartheta)) = \widehat{X}(\alpha) \vartheta + \xi,$$

where $\xi \in \Omega(P) \widehat{\otimes} \Gamma_k^\wedge$, so that $\widehat{X}(\alpha) \vartheta \in \text{im}(\widehat{X})$. Hence, \widehat{X} is surjective.

We prove the injectivity of \widehat{X} . Let us suppose that $w = \sum_k \alpha_k \omega^\wedge(\vartheta_k)$ belongs to $\ker(\widehat{X}) \setminus \{0\}$, where $\alpha_k \in \Omega(P) \otimes_M \mathfrak{hor}(P)$ and the elements $\vartheta_k \in \Gamma_{inv}^\wedge \setminus \{0\}$ are homogeneous and linearly independent.

This implies $\sum_k^* \widehat{X}(\alpha_k) \vartheta_k = 0$ where the summation is performed over indexes corresponding to the elements ϑ_k of the maximal degree. However, this is a contradiction, and hence \widehat{X} is injective. \square

Therefore, $\Omega(P)$ is a Hopf-Galois extension of $\Omega(M)$, at the level of graded-differential algebras [7]. The map \widehat{X} intertwines the corresponding differentials, and hence we have a cohomology isomorphism $H(\Omega(P) \otimes_M \Omega(P)) \leftrightarrow H(P) \otimes H(\Gamma^\wedge)$.

The inverse $\widehat{\tau}$ of \widehat{X} can be constructed explicitly in the following way. We have

$$\widehat{\tau}(\vartheta) = 1 \otimes \omega(\vartheta) - \sum_k \omega(\vartheta_k) \Delta_k \quad \widehat{\tau} \upharpoonright \mathcal{B} \otimes \mathcal{A} = \tau,$$

where $\sum_k \vartheta_k \otimes c_k = \varpi(\vartheta)$ and $X(\Delta_k) = c_k$. Using the above formulas, and the identity

$$\widehat{\tau}(w \otimes \alpha \beta) = w \sum_{\gamma \delta} (-1)^{\partial \alpha \partial \gamma} \gamma \widehat{\tau}(\alpha) \delta$$

where $\sum_{\gamma \delta} \gamma \otimes \delta = \widehat{\tau}(\beta)$, we can compute the values of $\widehat{\tau}$ on arbitrary elements.

4. CONCLUDING REMARKS

The proof of Proposition 1 is strongly based on the representation theory of compact matrix quantum groups. However, the compactness assumption figures only implicitly in the proof of Proposition 2. Actually, the proof works for general quantum structure groups. It is sufficient to ask that the bundle admits connections, that there exists a splitting ι and that the following natural decomposition holds

$$(6) \quad \Omega(M) \otimes_M \mathcal{B} \leftrightarrow \mathfrak{hor}(P) \leftrightarrow \mathcal{B} \otimes_M \Omega(M).$$

In the compact case the three assumptions hold automatically. Under the mentioned assumptions, if X is bijective then \hat{X} is bijective, too.

The first two assumptions ensure the existence of decompositions (3) and (4). The third assumption ensures that $\mathfrak{hor}(P)$ is a Hopf-Galois extension of $\Omega(M)$, if \mathcal{B} is a Hopf-Galois extension of \mathcal{V} . In other words, the restriction map (5) will be bijective. The rest of the proof is the same.

Proposition 2 holds also in the case when the higher-order calculus on the structure group is described by the corresponding [10] braided exterior algebra Γ^\vee . Because the general formalism of differential forms and connections can be formulated in the same way for braided exterior algebras.

Let $P = (\mathcal{B}, i, F)$ be an arbitrary quantum principal G -bundle over M . Let $\Omega(P)$ be an arbitrary differential structure describing the calculus on P . As explained in [5], the canonical bimodule sequence

$$0 \rightarrow \mathfrak{hor}^1(P) \rightarrow \Omega^1(P) \rightarrow \mathfrak{ver}^1(P) \rightarrow 0$$

is exact. Here $\mathfrak{ver}(P)$ is the graded-differential $*$ -algebra describing “verticalized” differential forms on the bundle. We have $\mathfrak{ver}(P) \leftrightarrow \mathcal{B} \otimes \Gamma_{inv}^\wedge$, at the level of graded left \mathcal{B} -modules

Let $\Omega_{hor}(P) \subseteq \mathfrak{hor}(P)$ be the $*$ -subalgebra generated by \mathcal{B} and $di(\mathcal{V})$. In the paper [1], these forms are called horizontal. The exactness condition introduced in [1] for the differential calculus can be formulated as the equality

$$(7) \quad \Omega_{hor}^1(P) = \mathfrak{hor}^1(P).$$

It is interesting to observe [2] that P is a Hopf-Galois extension if and only if (7) holds for *universal differential structures* (on both the bundle and the structure group). Therefore Proposition 1 can be reformulated as the statement that (7) always holds for completely universal differential calculus, if the structure group is compact.

On the other hand, for general differential structures the property (7) does not generally hold. For example, it is sufficient to consider a non-universal calculus Γ on G , and to assume that the calculus on P is universal. Another interesting phenomena is that even if (7) holds, generally $\Omega_{hor}(P) \neq \mathfrak{hor}(P)$ at the higher-order levels. A simple example for this is given by the trivial bundle $P = G$ over a one-point set M . If we assume that the calculus on the structure group G is described by the braided exterior algebra Γ^\vee and that $\Omega(P) = \Gamma^\wedge$, then

$$\Omega(M) = \mathbb{C} \iff \Gamma^\wedge = \Gamma^\vee.$$

5. ACKNOWLEDGEMENTS

It is a pleasure to acknowledge various interesting discussions with Profs Tomasz Brzeziński and Piotr Hajac, during Workshop on Quantum and Classical Gauge Theories, Stefan Banach International Mathematical Center, Warsaw. I would like to thank Profs Witold Kondracki and Robert Budzyński for kindly inviting me to visit Banach Center.

I am especially grateful to Prof Zbigniew Oziewicz, in particular for inviting me to visit the Institute of Theoretical Physics in Wrocław, and for his kind hospitality in this beautiful town. I would like to thank Prof Andrzej Borowiec for various interesting discussions in Wrocław.

REFERENCES

- [1] Brzeziński T & Majid S: *Quantum Group Gauge Theory on Quantum Spaces*, CMP **157**, 591–638 (1993)
- [2] Brzeziński T: *Geometrical Aspects of Quantum Group Gauge Theories*, Lecture, Workshop on Quantum & Classical Gauge Theories, Stefan Banach International Mathematical Center, Warsaw, Poland (May 1995)
- [3] Connes A: *Geometrie non commutative*, InterEditions Paris (1990)
- [4] Đurđević M: *Geometry of Quantum Principal Bundles I*, Preprint QmmP 6/92, Belgrade University, Serbia (To appear in CMP)
- [5] Đurđević M: *Geometry of Quantum Principal Bundles II—Extended Version*, Preprint, Instituto de Matematicas, UNAM, México (1994)
- [6] Đurđević M: *Quantum Principal Bundles & Tannaka-Krein Duality*, Preprint, Instituto de Matematicas, UNAM, México (1995)
- [7] Schauenburg P: *Hopf-Galois Extensions of Graded Algebras*, Preprint, Mathematisches Institut der Universität München, München, Germany (1994)
- [8] Schneider H J: *Hopf-Galois Extensions, Cross Products, and Clifford Theory*, Israel Journal of Mathematics
- [9] Woronowicz S L: *Compact Matrix Pseudogroups*, CMP **111** 613–665 (1987)
- [10] Woronowicz S L: *Differential Calculus on Compact Matrix Pseudogroups/ Quantum Groups*, CMP **122** 125–170 (1989)

INSTITUTO DE MATEMATICAS, UNAM, AREA DE LA INVESTIGACION CIENTIFICA, CIRCUITO EXTERIOR, CIUDAD UNIVERSITARIA, MÉXICO DF, CP 04510, MEXICO